Pisa 7 Marzo 2008

Problema del Cerchio:

$$u = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\left\{\frac{1}{2} + \sum_1^{\infty}\left(\frac{r}{R}\right)^n \cos n(\theta-\varphi)\right\} d\theta$$

$$\sum_1^{\infty} \rho^n \cos n\alpha = \sum_1^{\infty} \mathrm{Re}\,(\rho e^{i\alpha})^n = \mathrm{Re}\sum_1^{\infty} z^n$$

$$\rho e^{i\alpha} = \rho^n e^{in\alpha}$$

$$\mathrm{Re} \to \rho^n \cos n\alpha$$

$$\left[\begin{array}{l} z = \rho e^{i\alpha} \\ |z| = \rho = \frac{r}{R} < 1 \end{array}\right.$$

$$z\sum_1^{\infty} z^n = \frac{z}{1-z}$$

$$\mathrm{Re}\sum_1^{\infty} z^n = \mathrm{Re}\,\frac{\rho\, e^{i\alpha}}{1-\rho e^{i\alpha}} = \mathrm{Re}\,\frac{\rho e^{i\alpha} - \rho^2}{1+\rho^2 - 2\rho\cos\alpha}$$

$$= \frac{\rho\cos\alpha - \rho^2}{1+\rho^2-2\rho\cos\alpha}$$

$$= \frac{1}{2} + \frac{\rho\cos\alpha - \rho^2}{1+\rho^2 - 2\rho\cos\alpha} = \frac{1+\rho^2 - 2\rho\cos\alpha + 2\rho\cos\alpha - 2\rho^2}{2(1+\rho^2-2\rho\cos\alpha)}$$

$$u = \frac{1}{2\pi}\left(1-\frac{r^2}{R^2}\right)\int d\theta\, f(\theta)\frac{1}{1+\frac{r^2}{R^2} - 2\frac{r}{R}\cos\theta-\varphi}$$

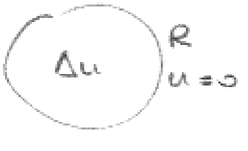
$$= \frac{R^2-r^2}{2\pi}\int f(\theta)\frac{1}{R^2+r^2-2rR\cos(\theta-\varphi)}\,d\theta = u(r,\varphi)$$

formula di Poisson.

$$\lim_{\substack{r\to R \\ \varphi\to\varphi_0}} u(r,\varphi) = f(\varphi_0)$$

Non omogeneo

$$\Delta u = F(r,\varphi) \qquad \varphi\in[0,2\pi]$$



$$u = \sum R(r)\phi(\varphi)$$

$$\Delta u = \sum R''\phi + \frac{1}{r}R'\phi + \frac{1}{r^2}R\phi'' \overset{!}{=} F(r,\varphi)$$

Cerchiamo primo e secondo membro come

$$\sum \phi(\varphi)\; c(r)$$

Il primo pezzo va bene. Cerchiamo

$$\phi'' = -\lambda\phi$$

$$\begin{cases} \phi(0) = \phi(2\pi) \\ \phi'(0) = \phi'(2\pi) \end{cases} \qquad \begin{array}{l}\lambda = n^2 \\ \phi = \cos(n\varphi),\quad \sin(n\varphi)\end{array}$$

$$\sum R_n'' \phi_n - \frac{1}{r} R' \phi_n - \frac{n^2}{r^2} R \phi_n \overset{!}{=} F(r,\varphi)$$

$$\equiv \sum F_n(r) \phi_n$$

E' come scrivere

$$\sum \phi_n L_n R_n = \sum \phi_n F_n(r)$$

I coeff. di Fourier devono essere uguali

$$\begin{cases} R_0'' + \frac{1}{R} R_0' = \frac{1}{2\pi} \int d\theta \, F(r,\theta) = C_0(r) \\ P_n'' + \frac{1}{r} P_n' - \frac{n^2}{r^2} = P_n = \frac{1}{\pi} \int_0^{2\pi} f(r,\theta) \cos(n\theta) \, d\theta \equiv c_n(r) \end{cases}$$

$$u = R_0 + \sum P_n \cos n\varphi + Q_n(r) \sin n\varphi$$

$$\Delta u = P_0'' + \frac{1}{r} P_0' + \sum_1^\infty \left(P_n'' + \frac{1}{r} P_n' - \frac{n^2 P_n}{r^2} \right) \cos n\varphi +$$

$$+ \sum \left(Q_n'' + \frac{1}{r} P_n' - \frac{n^2 Q_n}{r^2} \right) \sin n\varphi$$

$$\overset{!}{=} C_0(r) + \sum C_n(r) \cos n\varphi + S_n(r) \sin n\varphi$$

$P_0'' + \frac{1}{r} P_0' = C_0, \; P_0(R) = 0$ $\qquad$ $P_0'' + \frac{1}{r} P_0' \equiv L$ operatore

Sol. omog = $a + b \ln(R)$

$P_0 = A(r) + B(r) \log(r)$

2 funzioni a disposizione

1) Eq. non contenga derivate 2e

$$P_0' = (A' + B' \log r) + \frac{B}{2}$$

deve essere $A' + B' \log(r) \overset{!}{=} 0$

$$L(P_0) = \frac{B'}{r} + \text{(roba in cui B non è derivato)}$$

2) $\frac{B'}{r} = C_0$

$B(r) = \int_0^r C_0(x) \, x \, dx$ $\qquad$ per $B(r) \log(r)$ diff.bile in 0

$A' = - r C_0 \log(r)$

$A(R) + \log(R) B(R) = A(R) + \log(R) \int_0^R C_0(x) \, x \, dx = 0$ fissa A

$$A(r) = A(R) + \int_R^r A' = - \log(R) \int_0^R C_0 x \, dx + \int_r^R \log x \, C_0 x \, dx$$

$$P_0 = - \log(R) \int_0^R C_0 \, dx + \int_r^R \log x \, C_0 \, x \, dx + \log r \int_0^R C_0(x) \, x \, dx =$$

$$= \int_0^r x \, dx \left[\log \frac{r}{R} \right] + \int_r^R C_0 \, x \, dx \, \log \frac{x}{R}$$

$$P_0 = \int_0^r x\,dx\, C_0 \log\frac{r}{R} + \int_r^R C_0\, x\,dx \log\frac{x}{R}$$

$$P_n'' + \frac{1}{r} - \frac{n^2}{r^2} P_n = C_n$$

Cerco $P_n = A(r)\, r^n + B(r)\,\frac{1}{r^n}$

Impongo

$$P_n' = (\underset{0}{\underset{\overset{!}{\parallel}}{A' r^n + B'\frac{1}{r^n}}}) + nA\,r^{n-1} - \frac{nB}{r^{n+1}}$$

$$\begin{cases} A' r^n + B'\frac{1}{r^n} \overset{!}{=} 0 \\ nA' r^{n-1} - \frac{nB'}{r^{n+1}} \overset{!}{=} C_n \end{cases}$$ affinché l'eq. sia soddisfatta

$$I \cdot \frac{n}{r} + II$$

$$2n\,A'\frac{r^n}{r} = C_n \quad \Rightarrow \begin{cases} A' = \frac{1}{2n}\frac{C_n}{r^n}\, r \\ B' = -\frac{1}{2n}\, C_n\, r^n r \end{cases}$$

$$B(r) = -\frac{1}{2n}\int_0^r x\,dx\, C_n(x)\, x^n$$

$$A(R) = -\frac{1}{R^{2n}}\,\frac{1}{2n}\int_0^R x\,dx\, C_n x^n \quad \text{fisso } A$$

$$A(R) = \frac{1}{2n}\frac{1}{R^{2n}}\int_0^R x\,dx\, C_n(x)\, x^n - \int_r^R \frac{1}{2n}\,\frac{C_n}{x^n}\, x\,dx$$

$$P_n = \frac{1}{R^{2n}}\,\frac{1}{2n}\int_0^R x\,dx\, C_n \cdot (rx)^n - \frac{1}{2n}\int_r^R x\,dx\, C_n\left(\frac{r}{x}\right)^n +$$

$$-\frac{1}{2n}\int_0^r x\,dx\, C_n\left(\frac{x}{r}\right)^n$$

$$P_n = \frac{1}{2n}\int_0^r x\,dx\, C_n\left[\left(\frac{rx}{R^2}\right)^2 - \left(\frac{x}{r}\right)^n\right] + \frac{1}{2n}\int_r^R x\,dx\, C_n\left[\left(\frac{rx}{R^2}\right)^n - \left(\frac{r}{x}\right)^n\right]$$

$$= u_< + u_>$$

Pisa 12 Marzo 2008

Problema del Cerchio

$\Delta u = 0$ $$u(r,\varphi) = \rho_0(r) + \sum_{n=1}^{+\infty} P_n(r)\cos(n\varphi) + Q_n(r)\sin(n\varphi)$$

$u_<$, $u_>$

$u = R$

$$u = u_> + u_<$$

con $$u_< = \frac{1}{2\pi}\int_0^r x\,dx \int F(x,\theta)\,d\theta \;\ln\frac{r}{R} +$$

$$+ \frac{1}{\pi}\sum \frac{1}{2n}\int_0^r x\,dx \int F(x,\theta)\cos n\theta\,d\theta \left[\left(\frac{xr}{R^2}\right)^n - \left(\frac{x}{r}\right)^n\right]\cos n\varphi + \sin n\varphi$$

e $u_> = \int_r^R$

$$u_< = \frac{1}{2\pi}\int_0^r dS\; F(x,\theta)\;\ln\left(\frac{r}{R}\right) + \frac{1}{2\pi}\sum\frac{1}{n}\int_0^r dS\,F(x,\theta)\cos n(\theta-\varphi)\left[\left(\frac{xr}{R^2}\right)^n - \left(\frac{x}{r}\right)^n\right]$$

Ipotizzando che la serie converga si può scambiare Σ e $\int$

$$= \frac{1}{2\pi}\int dS\;F(x,\theta)\left[\ln\left(\frac{r}{R}\right) + \sum\frac{1}{n}\cos n(\theta-\varphi)\left[\left(\frac{xr}{R^2}\right)^n - \left(\frac{x}{r}\right)^n\right]\right]$$

$$\int_0^r \ldots = \lim_{r_0\to r}\int_0^{r_0}$$

$$\sum_1^\infty \frac{1}{n}\cos n\alpha\;\rho^n = \sum_1^\infty \cos n\alpha \int_0^\rho \rho^{n-1}\,d\rho =$$

$$= \int_0^\rho \frac{1}{\rho}\sum_1^\infty \rho^n \cos n\alpha\,d\rho$$

$\int_0^r$

$$= \int_0^\rho \frac{1}{\rho}\,\frac{\rho\cos\alpha - \rho^2}{1+\rho^2-2\rho\cos\alpha}\,d\rho$$

$$= \int_0^\rho \frac{\cos\alpha - \rho}{1+\rho^2-2\rho\cos\alpha}\,d\rho$$

$$= -\frac{1}{2}\int_0^\rho d\rho\;\frac{\frac{\partial}{\partial\rho}(1+\rho^2-2\rho\cos\alpha)}{1+\rho^2-2\cos\alpha\,\rho}$$

$$= -\frac{1}{2}\log(1+\rho^2-2\rho\cos\alpha$$

$$\int_0^{r_0} dS\,F \quad \ln 1 + \left(\frac{x}{r}\right)^2 - 2\frac{x}{r}\cos\alpha$$

$$\Rightarrow u_< = \frac{1}{2\pi}\int_0^r dS\,F(x,\theta)\left\{\frac{1}{2}\ln\left(\frac{r}{R}\right) - \frac{1}{2}\log\left[\frac{1+\frac{x^2r^2}{R^4} - \frac{2xr}{R^2}\cos(\theta-\varphi)}{1+\frac{x^2}{r^2} - \frac{2x}{r}\cos(\theta-\varphi)}\right]\right\}$$

$$= \frac{1}{4\pi}\int_0^r dS\,F(x,\theta)\;\ln\frac{r^2\left(1+\frac{x^2}{r^2} - 2\frac{x}{r}\cos(\theta-\varphi)\right)}{R^2\left(1+\frac{x^2R^2}{R^4} - 2\frac{xr}{R^2}\cos(\theta-\varphi)\right)} =$$

$$= \frac{1}{4\pi}\int_0^r dS\, F(x,\theta)\, \ln \frac{r^2+x^2-2xr\cos(\theta-\varphi)}{R^2+\frac{x^2r^2}{R^4}-2xr\cos(\theta-\varphi)}$$

che é invariante se scambio $x \to r$

$$\underset{u(r,\varphi)}{\underset{\parallel}{u_1+u_2}} = \underset{\substack{\text{cerchio} \\ \text{di raggio} \\ R}}{\iint} dS\, F(x,\theta)\, \ln \frac{r^2+x^2-2xr\cos(\theta-\varphi)}{R^2+\frac{x^2r^2}{R^2}-2xr\cos(\theta-\varphi)}$$

Eq. di Poisson del potenziale

$$\Delta u = F = -4\pi\rho \qquad (x,\theta) = \vec{r}_1$$

$$u(r,\varphi) = \int dS_1\, \rho(r_1)\, \log \frac{R^2+\frac{xr}{R^2}-2xr\cos(\theta-\varphi)}{x^2+r^2-2xr\cos(\theta-\varphi)}$$

$$= \int dS_1\, \rho(r_1)\, G(\underset{r,\varphi}{r_2}, \underset{x,\theta}{r_1}) \qquad \text{se} \quad \rho(\bar{r}_1) = \delta(\bar{r}_1 - \bar{r}_0)$$

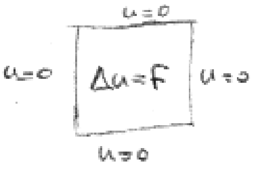
$$= G(\bar{r}_2, \bar{r}_0)$$

Chi é $G(\bar{r}_2, \bar{r}_0)$?

È la funzione di Green: il potenziale in $\bar{r}_2$ quando é presente una carica $q=1$ in $\bar{r}_0$ con la condizione $G(r_2, r_0)$ se r_2 é sulla circonferenza

Si ha la proprietà $G(\bar{r}_1, \bar{r}_2) = G(\bar{r}_2, \bar{r}_1)$

Problema del quadrato



$$u = \sum XY$$

$$\Delta u = X''Y + Y''X = F(x,y)$$

Leggo I e II membro come $\sum X\, \underset{\text{coeff. di } y}{c(y)}$

Richiedo $X'' = -\lambda X$

$X(0) = X(\pi) = 0$

$$\sum X(Y''-Y\lambda) = F(x,y) = \sum F_n(y) X_n$$

$$\begin{cases} Y_n'' - n^2 Y_n = F_n(y) \\ Y_n(0) = Y_n(\pi) = 0 \end{cases}$$

$$\begin{cases} A^{*}(y)\, e^{ny} + B(y)\, e^{-ny} = 0 \\ A' e^{ny} + B' e^{-ny} = 0 \end{cases}$$

$$y' = A n e^{ny} - B n e^{-ny}$$

$$A' n e^{ny} - B' e^{-ny}\, n \;= F_n$$

Equazione delle onde

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = F(x,t) \qquad [0,\ell]$$

$u(x,0) = f(x)$

$\frac{\partial u}{\partial t}(x,0) = g(x)$

$u(0,t) = a(t)$

$u(\ell,t) = b(t)$

eq. lineare $\Rightarrow$ cerco u come somma di

$$u = u_F + u_f + u_g + u_a + u_b$$

In cui u_f è la soluzione con solo $u_f \neq 0$

Per separazione di variabili: pongo $u_f \equiv u$

$$\frac{1}{c_2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

$\dot{u}(x,0) = 0 \quad , \; u(x,0) = f \qquad u = 0$ agli estremi

Sol. part. $X(x)\,T(t)$

$$u = \Sigma\, XT$$

$$\frac{1}{c^2} X T'' = X'' T \qquad \frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = -\lambda$$

$$X'' + \lambda X = 0 \qquad X(0) = 0 = X(\ell)$$

$$X_n = \sin\frac{n\pi x}{\ell} \qquad \Rightarrow \quad \lambda = \frac{n^2\pi^2}{\ell^2}$$

$$T'' + \frac{n^2\pi^2 c^2}{\ell^2} T = 0$$

$$T_n = \cos\frac{n\pi c t}{\ell}$$ [illegible]

$$u = \Sigma\, a_n \cos\frac{n\pi c t}{\ell} \sin\frac{n\pi x}{\ell}$$

$$u(x,0) = \Sigma\, a_n \sin\frac{n\pi x}{\ell} = f(x) = \Sigma\, f_n \sin\frac{n\pi x}{\ell}$$

$$= \frac{1}{2}\Sigma\, f_n \left\{ \sin\frac{n\pi(x+ct)}{\ell} + \sin\frac{n\pi(x-ct)}{\ell} \right\}$$

$$= \frac{1}{2}\Sigma\, f_n \sin\frac{n\pi}{\ell}(\xi)$$

$$u = \frac{1}{2}\Big[\underset{\uparrow}{f}(x+ct) + f(x-ct)\Big]$$

prolungamento periodico dispari di f

Pisa 12 Marzo 2008 ESERCITAZIONE

$L^2(0,1)$ $\sin n\pi x^2$ completo ?

$\int_0^1 f \sin n\pi x^2 \, dx = 0 \quad \forall n \Rightarrow ? \quad f = 0$

$\sin n\pi z$ è compl in $\mathbb{L}^2[0,1]$

$z = x^2 \qquad 0 = \int_0^1 f(x(z)) \sin n\pi z \, \frac{dz}{\sqrt{z}}$

Se $\frac{f(x(z))}{\sqrt{z}} \in \mathbb{L}^2[0,1]$ allora $f(x) = 0$

So che $\int_0^1 |f(x)|^2 dx < \infty$

$$\int_0^1 dz \, \frac{1}{z} |f(x(z))|^2 = \int_0^1 2x \, dx \, \frac{1}{x^2} |f(x)|^2 = 2 \int_0^1 \underset{\underset{\notin \mathbb{L}^2 \ \forall f}{\nearrow}}{\frac{|f(x)|^2}{x}} dx$$

Tentativo fallito.

$g \in \mathbb{L}^1 \quad (\mathbb{L}^2 \subset \mathbb{L}^1)$

$\int_0^\pi \sin nx \; g(x) \, dx = a_n$

Supponiamo $a_n = 0 \ \forall n$. È $g = 0$ q.o. ?

Se $\cancel{\|g\|_1 =} A \sum |a_n|$ sì, ma questo non è vero!

In $\mathbb{L}^2$ $\|g\|_2^2 = \sum |a_n|$

$g \in L^1 \quad$ coeff $= 0 \Rightarrow g = 0$

$$0 = \int_0^\pi g(x) \sin nx \, dx = \cancel{\underset{\underset{G(x)}{\shortparallel}}{\int_0^x g\,dt} \; \underset{\underset{0}{\shortparallel}}{\sin nx\Big|_0^\pi}} - \int_0^\pi G(x) \, n \cos nx \, dx \qquad \forall n > 0$$

$\int_0^\pi G(x) \, dx \cos nx = 0$

$G(x) = \int_0^x g(t) dt$ è continua in x e derivabile q.o e

$G' = f$ q.o.

$$G(x) = a_0 + \underset{\underset{0 \text{ per ipotesi}}{\shortparallel}}{\sum a_n \cos(nx)}$$

$\Rightarrow G = a_0$ costante.

$G(x) = \int_0^x g(t)\,dt$

$\frac{dG}{dx} = g = 0$ q.o

Se $g \in \mathbb{L}^1$ $\int_0^\pi g \cos nx \, dx = 0$ $\forall n$

Quindi $\frac{f(x(z))}{\sqrt{z}} = 0 \Rightarrow f(x) = 0$

perché $f \in \mathbb{L}^2 \subset \mathbb{L}^1$ e per quanto visto i coeff di f sono nulli

$\Rightarrow f = 0$ in $\mathbb{L}^1$.

$\int_0^\pi \frac{1}{\sqrt{z}} |f(x(z))| \, dz =$ $z = x^2$ $\frac{dz}{\sqrt{z}} = \frac{2dx}{-x}$

$= 2 \int_0^\pi dx \, |f(x)| < \infty$ e tanto basta

$0 = \int_0^1 f(x) \sin n\pi x' dx = \int_0^x f dt \sin n\pi x \Big|_0^1 - \int_0^1 F(x) n\pi - 2x \cos n\pi x^2 dx$

$n > 1$

$z = x^2$ $x = \sqrt{z}$

$\int_0^1 \underline{F(\sqrt{z})\sqrt{z}} \cos n\pi z \underline{\frac{dx}{dz}} dz = 0$

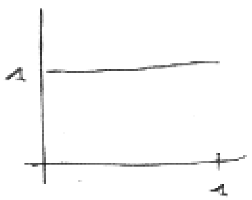
Se $F(\sqrt{z})$ é $\mathbb{L}^2 \Rightarrow F(\sqrt{z}) = \text{cost}$

$F(x) = \text{cost}$

$\int_0^x f dt = \text{cost}$

$F(\sqrt{z})$ continua $\Rightarrow \mathbb{L}^2$

$F(x)$ continua $\not\Rightarrow$



$= \frac{4}{\pi} \Sigma \frac{1}{2n+1} \sin(2n+1) dx$

Perché solo i dispari?

Perché $f = 1$ é simm $\sim \frac{\pi}{2}$

$\sin(2k\pi x)$ sono anti

$\Sigma(\pi + 1) = -1$ (si ottiene il prolungamento dispari)

$$\int_0^x 1\,dx = x = \frac{4}{\pi}\sum \frac{1}{2k+1}\int_0^x \sin(2k+1)t\;dt$$

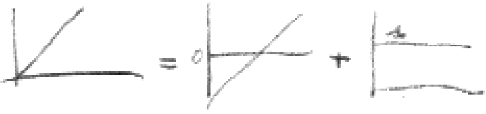
$$= \frac{4}{\pi}\sum \frac{1}{(2k+1)^2}(-1)\left[\cos(2k+1)x \;-1\right]$$

$$= \frac{4}{\pi}\sum \frac{1}{(2k+1)^2} - 4\pi\sum \frac{1}{(2k+1)^2}\cos(2k+1)x$$

$$\frac{\pi}{2} = \frac{4}{\pi}\sum \frac{1}{(2k+1)^2} \qquad \sum \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

$$x = \frac{\pi}{2} - \frac{4}{\pi}\sum \frac{1}{(2k+1)^2}\cos(2k+1)x$$

perche



$$(x,1) = \frac{4\pi}{\pi}\sum \frac{1}{(2k+1)^2}$$

$$\int_{\pi/2}^x 1\,dx = \cancel{x} - \frac{\pi}{2}$$

$$= \frac{4}{\pi}\sum \frac{1}{2k+1}\int_{\pi/2}^x \cos(2k+1)t$$

$$= -\frac{4}{\pi}\sum \frac{1}{(2k+1)^2}\cos(2k+1)\cancel{x}$$

$$\sum_1^\infty \frac{1}{n^2} = \sum_1 \frac{1}{(2k)^2} + \sum_0 \frac{1}{(2k+1)^2}$$

$$= \frac{1}{4}\sum_1 \frac{1}{n^2} + \frac{\pi^2}{8}$$

$$\frac{3}{4}\sum \frac{1}{n^2} = \frac{\pi^2}{8}$$

$$\sum \frac{1}{n^2} = \frac{\pi^2}{6}$$

Pisa 14 Marzo 2008

Convergenza debole

Def: In $\mathbb{L}^2$ $\quad f_n \xrightarrow{deb} \underset{\mathbb{L}^2}{\overset{\cap}{f}}$ se $\forall g \in \mathbb{L}^2$

$$(f_n, g) \to (f, g)$$

Se $f_n \xrightarrow{\| \, \|} f$ $\quad |(f_n - f, g)| \leq \|f_n - f\| \|g\| \to 0$

$$\Rightarrow \quad \| \, \| \Rightarrow deb$$

$deb \not\Rightarrow \| \, \|$

$\sin(nx) \xrightarrow{deb} 0 \quad$ in $\mathbb{L}^2(0,\pi)$

$g \qquad \int_0^\pi \sin(nx)\, g \to 0 \qquad$ per Riemann Lebesgue

o per Bessel

$\sum |(\sin(nx), g)|^2 \frac{\pi}{2} = \|g\|$

Vale per $\{e_n\}$ ortonormali $\qquad$ Bessel $\{\sum |(e_n, x)|^2 \leq \|x\|^2$

$e_n \xrightarrow{deb} 0$

se $\|f_n\| \to \|f\|$ $\quad$ allora $f_n \xrightarrow{\| \|} f$

Se $f_n \xrightarrow{deb} f$

allora $\|f_n\| < M$

$\mathbb{L}^2(0, \infty) \qquad f \in \mathbb{L}^2$

$f_n = n^\alpha f(nx)$

$\|f_n\|^2 < M \qquad n^{2\alpha} \int_0^\infty |f(nx)|^2 \, dx =$

$$= n^{2\alpha - 1} \int_0^\infty |f(z)|^2 dz$$

$n^{2\alpha - 1} < M$

Se $2\alpha - 1 > 0$ $\quad$ nessun limite

Se $2\alpha - 1 < 0$ $\quad \|f_n\| < 0 \Rightarrow \|f_n\| \to 0$ e quindi conv. in $\| \, \|$

Se $2\alpha - 1 = 0$

$f_n = \sqrt{n}\, f(nx) \xrightarrow{deb} 0$

Verifico:

$$\sqrt{n} \int_0^\infty f(nx)\, g(x)\, dx =$$

$$= \sqrt{n} \int_0^\delta f(nx)\, g(x)\, dx + \sqrt{n} \int_\delta^\infty f(nx)\, g(x)\, dx$$

$$\mathrm{I} \qquad + \qquad \mathrm{II}$$

$$|\mathrm{I}|^2 \leq n \int_0^\delta |f(nx)|^2 dx \int_0^\delta |g|^2 dx$$

Schwarz

$$= \int_0^{n\delta} |f(x)|^2 dx \int_0^\delta |g(x)|^2 dx$$

$$\leq \|f\|^2 \int_0^\delta |g(x)|^2 dx < \varepsilon$$

per qualche δ appropriato indip da n.

$$|\mathrm{II}|^2 = n \left| \int_\delta^\infty f(nx)\, g(x)\, dx \right|^2$$

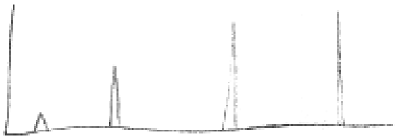
$$\leq n \int_\delta^\infty |f(nx)|^2 dx\, \|g\|^2 = \int_{n\delta}^\infty |f(x)|^2 \|g\|^2$$

per $n \to \infty$ $\int_{n\delta}^\infty |f(x)|^2 \|g\|^2 < \varepsilon$

$\sqrt{n}\, f(nx) \overset{\|\ \|}{\not\longrightarrow} 0$ No

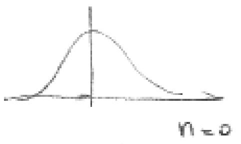
Stesso vale per $\mathbb{L}^2(-\infty, \infty)$

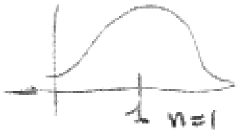
! $f \in \mathbb{L}^1(-\infty,\infty) \not\Rightarrow f \longrightarrow 0$ per $x \to \infty$



$\mathbb{L}^2(-\infty,\infty)$, f

$f_n = f(x-n)$





$$\|f_n\|^2 = \int_{-\infty}^{+\infty} |f(x-n)|^2 dx = \|f\|^2$$

$$\int_{-\infty}^{+\infty} f(x-n)\, g(x)\, dx = \int_{-R}^{R} f(x-n)\, g(x) + \int_{|x|>R} f(x-n)\, g(x)\, dx$$

$$\mathrm{I} \qquad\qquad \mathrm{II}$$

$|\mathrm{II}|^2 < \varepsilon$ per R opportuno
indip da R

$$\left| \int_{|x|>R} f(x-n) g\, dx \right| \leq \int_{|x|>R} |f(x-n)|^2 \int_{|x|>R} |g|^2 dx.$$

$$\leq \|f\|^2 \cdot \int_{|x|>R} |g|^2 dx < \varepsilon$$

per qualche R suff. grande

$$\left|\int_{-R}^{R} f(x-n)\, g\, dx\right|^2 \leq \int_{-R}^{R} |f(x-n)|^2 dx \ \|g\|^2$$

$z = x-n$

$$\int_{-R+n}^{R+n} |f(x)|^2 dx \ \|g\|^2 \leq \int_{-R+n}^{+\infty} |f(x)|^2 dx \ \|g\|^2 < \varepsilon$$

per $n \to \infty$

$f(x-n) \overset{\|\,\|}{\not\to} 0$ se $\exists$ deve coincidere con il lim deb.

$\mathbb{L}^2 (0,\pi)$

$\{x \sin(nx)\}$ Entrambi completi

$\{x^2 \sin(nx)\}$

$\int f \, x \sin nx \, dx = 0$

$f \in \mathbb{L}^2 \Rightarrow xf \in \mathbb{L}^2$ $\quad \sin nx$ compl $\Rightarrow f \equiv 0$

Stesso per $x^2 \sin nx$

Tolgo $n=1$ $\quad$ Controllo quelli che ho tolto

$\int (f\,x) \sin nx \, dx = 0$

$\overset{\shortparallel}{g} = a_1 \sin x + a_2 \sin 2x + \dots$ $\qquad g = xf = a \sin x$

$f = a \frac{\sin x}{x}$ $\quad a \neq 0$

$\Rightarrow$ non è più compl perché $a \frac{\sin x}{x}$ è lim.

$\int (x^2 f) \sin nx \, dx = 0$

$x^2 f = a \sin x$

$f = a \frac{\sin x}{x^2} \in \mathbb{L}^2 \Leftrightarrow a = 0 \Leftrightarrow f = 0 \Rightarrow$ completo

$\quad \overset{\wr\wr}{1/x \notin \mathbb{L}^1}$

in un intorno di $x=0$

Tolgo i primi due

$xf = a \sin x + b \sin 2x$

$f = a \frac{\sin x}{x} + b \frac{\sin 2x}{x} \in \mathbb{L}^2$

$x^2 f = a \frac{\sin x}{x^2} + b \frac{\sin 2x}{x^2} \in L^2$ senza che $a, b = 0$

$A\left(2\frac{\sin x}{x^2} - \frac{\sin 2x}{x^2}\right) \neq 0$ $\Rightarrow$ non è più completo

lim

Provare con $x^3 \sin(nx)$

$\boxed{\Delta u = 0}$ $\qquad n \cdot \vec{\nabla} u \equiv \frac{\partial u}{\partial n}$

$\frac{\partial u}{\partial n} = f$

derivata normale

0) Sol. ∃ sol. se

$$\int_0^{2\pi} d\theta\, f(\theta) = 0$$

$$\int_{\text{Cerchio}} \underset{0}{\underset{\parallel}{\operatorname{div} \nabla u}} \cdot dS = \int_{Rc} n \cdot \nabla n \, d\ell = \int_{Rc} f \, R \, d\theta$$

$$f = \sum_1^\infty c_n \cos n\varphi + d_n \sin n\varphi \qquad f_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta$$

$$u = a_0 + \sum_n a_n r^n \cos n\varphi + b_n r^n \sin n\varphi$$

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial r}\bigg|_{r=R} = \sum a_n n R^{n-1} \cos n\varphi + b_n n R^{n-1} \sin n\varphi \overset{!}{=} f$$

$$a_n n \frac{R^n}{R} = c_n \qquad b_n n \frac{R^n}{R} = d_n$$

$$u = \sum_1^\infty \frac{R}{n}\left(\frac{r}{R}\right)^n \left[c_n \cos n\varphi + d_n \sin n\varphi\right]$$

$$c_n = \frac{1}{\pi}\int f(\theta) \cos\theta \qquad d_n = \frac{1}{\pi}\int f(\theta) \sin n\theta$$

$$[\cdots] = \frac{1}{\pi}\int f(\theta) \cos n(\theta - \varphi)\, d\theta$$

$$u = \frac{R}{\pi} \sum_1^\infty \frac{1}{n}\left(\frac{r}{R}\right)^n \int_0^{2\pi} f(\theta) \cos n(\theta - \varphi)\, d\theta =$$

$$= \frac{R}{\pi}\int f(\theta) \sum \frac{1}{n}\left(\frac{r}{R}\right)^n \cos n(\theta - \varphi)$$

$$\sum_{1}^{\infty} \frac{1}{n} \rho^n \cos n\alpha = -\frac{1}{2} \log \left[1 + \rho^2 - 2\rho\cos\alpha \right]$$

$$u = -\frac{R}{2\pi} \int f(\theta) \log \left[1 + \frac{r^2}{R^2} - 2\frac{r}{R} \cos(\theta - \varphi) \right] d\theta$$

$$= -\frac{R}{\pi} \int f(\theta) \log \left[\frac{R^2 + r^2 - 2rR\cos(\theta - \varphi)}{R^2} \right.$$

$$u = a_0 - \frac{R}{2\pi} \int_0^{2\pi} f(\theta) \log \left[R^2 + r^2 - 2rR \cos(\theta - \varphi) \right] d\theta$$